

Doubled geometry and T-folds

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ABSTRACT: The doubled formulation of string theory, which is T-duality covariant and enlarges spacetime with extra coordinates conjugate to winding number, is reformulated and its geometric and topological features examined. It is used to formulate string theory in T-fold backgrounds with T-duality transition functions and a quantum implementation of the constraints of the doubled formalism is presented. This establishes the quantum equivalence to the usual sigma-model formalism for world-sheets of arbitrary genus, provided a topological term is added to the action. The quantisation involves a local choice of polarisation, but the results are independent of this. The natural dilaton of the doubled formalism is duality-invariant and so T-duality is a perturbative symmetry for the perturbation theory in the corresponding coupling constant. It is shown how this dilaton is related to the dilaton of the conventional sigma-model which does transform under T-duality. The generalisation of the doubled formalism to the superstring is given and shown to be equivalent to the usual formulation. Finally, the formalism is generalised to one in which the whole spacetime is doubled.

KEYWORDS: Flux compactifications, Superstring Vacua, String Duality.

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1. Introduction

A conventional ‘geometric’ string background consists of a spacetime manifold equipped with a metric and various gauge fields, which may be connections for bundles or gerbes over spacetime, and satisfying field equations arising from the requirement that quantising the corresponding sigma-model gives a conformal field theory. However, string theory can be consistently defined in many non-geometric backgrounds that are not of this type [1]–[12], and it seems likely that generic string theory solutions will be non-geometric. In particular, conventional compactifications can be generalised to ones where the internal compact manifold is replaced with string theory in a non-geometric background, resulting in a conventional theory in a geometric four-dimensional spacetime. This has been explored in [2], where it was argued that this gives a much wider class of effective four-dimensional field theories than can be obtained from conventional compactifications.

An important class of non-geometric backgrounds are those which are constructed from local patches, each of which is a patch of a conventional geometric string background, but these patches are glued together with transition functions that include duality transformations as well as the usual diffeomorphisms and gauge transformations [1]. This can give

T-folds with T-duality transition functions or U-folds with U-duality transition functions, or mirror-folds with mirror symmetry transition functions. T-folds or U-folds require each patch to be the product of a torus with some open set in a base space N , so that the T-fold has a torus fibration over N , while a mirror-folds have a Calabi-Yau fibration. More exotic possibilities include gluing a heterotic string theory patch with a T^4 fibration to a IIA string theory patch with a $K3$ fibration, as these theories are dual [14].

The T-fold backgrounds can be studied within perturbative string theory and so can be most fully treated. Locally, a T-fold looks like a conventional patch of a spacetime with a torus fibration. T-duality [16] was shown in [17–20] to be a symmetry of spacetimes that torus fibrations in which there was a $U(1)^d$ isometry, so that they are principle $U(1)^d$ bundles. This was generalised in [10] to the case of general torus bundles in which there may be no globally defined killing vectors, so establishing the result that T-duality can be done fibrewise, provided that certain obstructions are absent. However, applying T-duality to geometric backgrounds with fluxes in general gives a T-fold [1], not a geometric space, and so one is led to consider such backgrounds.

Let X^i be coordinates on the torus fibres, and Y^m be the remaining coordinates, and the d^2 moduli $\tau \in O(d, d)/O(d) \times O(d)$ of the torus T^d depend on Y in general. Quantising the coordinates X gives a torus conformal field theory specified by the moduli $\tau(Y)$ for each Y . The conformal field theory has an $O(d, d; \mathbb{Z})$ symmetry, and moduli related by an $O(d, d; \mathbb{Z})$ transformation determine the same conformal field theory. Then $O(d, d; \mathbb{Z})$ transition functions allow the consistent construction of a bundle of torus conformal field theories over some base space N with local coordinates Y . One can then integrate over the fields Y^m to give the quantum string theory in such a T-fold background.

In formulating the conformal field theory on the T^d fibres, an extra d coordinates \tilde{X}_i for a dual torus \tilde{T}^d are needed. These are conjugate to the winding number, and are needed to write vertex operators such as $e^{ik_L \cdot X_L}$ where $X_L = X - \tilde{X}$, and to formulate string field theory. For string field theory in toroidal backgrounds, the string field should depend explicitly on \tilde{X} as well as X [21]. This means that generic solutions of string field theory depend on both X and \tilde{X} ; some interesting examples of backgrounds depending non-trivially on \tilde{X} have been investigated in [2]. However, T-fold backgrounds do not depend explicitly on \tilde{X} , so can be expressed in terms of conventional spacetime fields locally. In [1], a formulation of string theory on a T-fold was given, with a target space which had a T^{2d} doubled torus fibration with local coordinates X^i, \tilde{X}_i, Y^m . For a T-fold, the doubled patches fit together to form a T^{2d} bundle \hat{M} over the base N , and the theory is formulated as a sigma-model with target space \hat{M} . This formulation is manifestly $O(d, d; \mathbb{Z})$ invariant. To obtain the conventional theory, a constraint is imposed that halves the doubled degrees of freedom on the torus; for a flat background, this constraint requires half of the $2d$ scalar fields on T^{2d} to be left-movers and half right-movers. The constraint is well-defined on \hat{M} and is $O(d, d; \mathbb{Z})$ invariant. The conventional theory is regained by choosing a polarisation, i.e. by choosing half of the coordinates on the torus T^{2d} to be the physical spacetime coordinates. This involves choosing a $T^d \subset T^{2d}$ and can be done globally for a geometric background, but only locally in each patch for a T-fold, and in general the polarisation changes from patch to patch. T-duality can be thought of as acting

to change the polarisation [1], and so the statement that the physics is T-duality invariant implies that the choice of polarisation does not affect the physics.

The sigma-model on the doubled space \hat{M} can be quantised in the usual way, but the problem arises as to how to implement the constraint. One approach is to first quantise the variables X, \tilde{X} , for fixed Y . One can first solve the constraint and then quantise. The constraint is a self-duality condition that relates $\partial\tilde{X}$ and ∂X , and it is important that in the doubled formulation for a T-fold, \tilde{X} only enters through its derivative $\partial\tilde{X}$. Then the constraint can be used to give $\partial\tilde{X}$ in terms of ∂X . The constraint implies the classical world-sheet field equations for X, \tilde{X} , and for a cylindrical world-sheet the field equation for X can be solved in terms of the oscillators, momenta and winding modes for X . These can be quantised in the usual way to obtain the usual CFT on T^d . This gives a torus CFT with moduli $\tau(Y)$ for each point Y and hence a bundle of CFT's over N . The final step is then to quantise Y .

While this paper was in preparation, the paper [13] appeared, giving a constrained Hamiltonian approach for T-folds on cylindrical world-sheets, using Dirac brackets to quantise the system. This was applied to an example of a T-fold which is an asymmetric orbifold, and gave the same results as the conventional quantisation of this system. An interesting feature is that, at least for this explicit example, no choice of polarisation is needed.

However, it is desirable to have an off-shell formulation which does not impose field equations, and which applies to world-sheets that are Riemann surfaces of arbitrary genus. The constraint requires that a certain conserved current J vanishes, and it was suggested in [1] that this could be imposed by gauging the symmetry generated by J , adding a coupling $C \cdot J$ to a world-sheet gauge field, plus quadratic terms in C . It will be shown here that this does not quite work, but that one can instead gauge half of the currents J and this is sufficient to impose the constraint. Gauge-fixing and integrating out the gauge fields then recovers the usual (undoubled) sigma-model formulation locally. The choice of which half of the currents J to gauge is the choice of polarisation. For a geometric background, there is a global choice of polarisation and the usual formulation is recovered, but for a T-fold, there is no global choice, and the quantisation involves a choice of a different polarisation in each patch. Nonetheless, the resulting quantum theories should patch together to give a consistent well-defined theory. Then the situation is similar to gauge theory, which has a globally-well defined gauge-invariant quantum effective action, even though in the calculation of this one must gauge-fix, breaking the manifest symmetry, and in general one must make a different gauge choice in different patches.

This allows the definition of the quantum theory for Riemann surfaces of arbitrary genus, and it is found that the classical action must be supplemented by a topological term in order to achieve complete equivalence to the usual formulation. This term does not affect the classical theory, but introduces certain relative signs in the sum over topological sectors. It is also shown that there is a functional Jacobian that arises in changing between the formulations, and this has important physical consequences at one-loop and higher.

For a T-fold to be a good string background, the resulting quantum theory must be conformal and modular invariant. Conformal invariance requires that in any patch, g, b and the dilaton must satisfy the usual β -function equations, so that there is a conformal field

theory in each patch. Modular invariance then imposes conditions on the allowed transition functions. For example, a special class of T-folds are asymmetric orbifolds, and it is well-known that modular invariance only allows a restricted class of asymmetric orbifolds. Then a T-fold string background is locally conformal, i.e. it is constructed from patches in each of which the geometric data satisfies β -function equations, and the transition functions are chosen to be compatible with modular invariance.

In addition to showing how to quantise in the doubled formalism and establishing its equivalence to the usual formalism, a number of other issues left over from [1] will be discussed. The doubled formulation will be re-expressed in a form in which its geometric structure is more apparent, using results from [10]. A careful treatment of the global structure will be given and applied to the quantum theory. A puzzle arises in the issue of the dilaton coupling. In the doubled formalism, the natural dilaton coupling through a Fradkin-Tseytlin term is necessarily duality invariant, while it is known that the dilaton in the usual sigma-model transforms under duality. It will be shown that these results are consistent and that the dilatons in the two formalisms are indeed different, and the relationship between them will be found. The string perturbation theory involving the dilaton arising in the doubled formalism is duality invariant so that T-duality is manifestly a perturbative symmetry, and this coupling constant is the same as that of string field theory [21].

In section 8, the results are extended to the supersymmetric doubled formalism, and the relationship to the usual formalism again established. In section 10, the formalism is generalised to one in which all coordinates are doubled, not just the tori, and this gives a formalism applicable to general spaces, not just to torus bundles.

There is an interesting relation with Hitchin's generalised geometry [15]. In generalised geometry, a conventional geometry with a D -dimensional manifold M equipped with a metric tensors g and a gerbe connection b is considered, and it is found that many features are elegantly expressed on $T \oplus T^*(M)$, or the twisting of this by a gerbe, and there is a natural action of the continuous group $O(d, d)$. The transition functions are diffeomorphisms and b -field gauge transformations, giving transition functions $GL(D, \mathbb{R})$ on $T \oplus T^*(M)$, or the semi-direct product of this with b -transformations for the twisted version. T-folds are more general than generalised geometry, with transition functions including the discrete group $O(d, d; \mathbb{Z})$ for T^d fibrations, and are not manifolds with tensor fields g, H . While generalised geometry doubles the tangent space, doubled geometry doubles the torus fibres, or the whole manifold. Doubling the manifold of course entails doubling the tangent space. Both kinds of geometry have a natural action of $O(d, d)$ and similar $O(d, d)$ covariant structures appear in both. However, doubled geometry is governed by the discrete group $O(d, d; \mathbb{Z})$ and T-duality is an essential feature, while in generalised geometry only the continuous group $O(D, D)$ appears. On the other hand, the generalised geometry approach can be applied to any manifold, while T-folds arise naturally only for torus fibrations. The relation between doubled geometry and generalised geometry will be discussed further elsewhere.

2. String backgrounds

The string backgrounds that will be considered here can be constructed from local patches and in each patch there is a conventional string background, so that each patch is diffeomorphic to a contractible open set in \mathbb{R}^D equipped with a metric g and a 2-form b . A geometric background is a manifold made from patches of this type with transition functions that are diffeomorphisms and 2-form gauge transformations $\delta b = d\lambda$, so that g and $H = db$ are tensor fields on M . T-folds are non-geometric backgrounds where the transition functions also include T-dualities, so that the result is not a manifold with tensor fields. In this section, the local structure of such backgrounds will be reviewed, and the global structure will be discussed in section 7.

A geometric string background is then a manifold M with a metric g and closed 3-form H . In each local patch, one can introduce local coordinates ϕ^μ ($\mu, \nu = 1, \dots, D$, where D is the dimension of M) and H is given in terms of a 2-form potential b , $H = db$. The lagrangian is

$$L = \frac{1}{2}g_{\mu\nu}d\phi^\mu \wedge *d\phi^\nu + \frac{1}{2}b_{\mu\nu}d\phi^\mu \wedge d\phi^\nu \tag{2.1}$$

Here $d\phi$ is a 1-form on M pulled-back to the world-sheet. The world-sheet metric is taken to have Lorentzian signature, and $*$ is the world-sheet Hodge duality operator satisfying $(*)^2 = 1$. (The formulae will be presented here for Lorentzian world-sheet metrics. The continuation to Euclidean signature is straightforward, and in most formulae in this paper is given by replacing $*$ with $-i*$, as $(-i*)^2 = 1$ in Euclidean signature, and taking lagrangian 2-forms $L \rightarrow -iL$. In (2.1), this has the net effect of replacing b with ib .)

If M is a torus bundle over some base manifold N with fibres T^d , then it can be constructed from patches of the form $U' = U \times T^d$ where U is a patch on the base manifold N , diffeomorphic to a contractible open set in \mathbb{R}^{D-d} . In each such patch U' , there are d commuting vector fields $k_i = k_i^\mu \partial / \partial \phi^\mu$ tangent to the fibres, with each k_i generating a periodic orbit. It will be assumed they are Killing vectors with

$$\mathcal{L}_i H = 0 \tag{2.2}$$

where \mathcal{L}_i is the Lie derivative with respect to k_i , generating a freely acting $U(1)^d$ isometry of U' . For principle bundles, these extend to globally defined Killing vector fields on M , but for general torus bundles they do not. In [10], T-duality and the gauging of sigma-models was generalised to such general torus bundles without isometries.

Consider then a patch of a string background $U' = U \times T^d$ with a metric g , a 2-form b and d Killing vectors in U' tangent to the fibres. They could fit together to form either a torus bundle over N , or a T-fold over N . The norm of the Killing vectors

$$G_{ij} = g(k_i, k_j) \tag{2.3}$$

defines a matrix of functions on U and, as this is non-degenerate in U (assuming g restricted to the fibres is positive definite), there are one-forms ξ^i with components

$$\xi_\mu^i = G^{ij} g_{\mu\nu} k_j^\nu \tag{2.4}$$

dual to the Killing vectors. The field strengths

$$F^i = d\xi^i \tag{2.5}$$

satisfy

$$\iota_i F^j = 0 \tag{2.6}$$

where ι_i denotes contraction with k_i . The metric can be written as

$$g = \bar{g} + G_{ij} \xi^i \otimes \xi^j \tag{2.7}$$

where \bar{g} is a metric on U . The ξ^i define a natural frame on the fibres over U .

Next we give an alternative derivation the results of [10] for the general form of H . The condition (2.2) implies that $\iota_{j_1} \dots \iota_{j_n} H$ is closed for $n = 1, 2, 3$, so that in a contractible open set $V \subset M$ they are exact. Then

$$\iota_i \iota_j \iota_k H = K_{ijk} \tag{2.8}$$

are constants (in V) and

$$K = \frac{1}{6} K_{ijk} dX^i \wedge dX^j \wedge dX^k \tag{2.9}$$

defines a closed 3-form, so that

$$H' = H - K \tag{2.10}$$

is closed and satisfies

$$\iota_i \iota_j \iota_k H' = 0 \tag{2.11}$$

The analysis of [10] can now be applied to H' . In V , there is are 1-forms v_i and 0-forms $B_{ij} = -B_{ji}$ such that

$$\iota_i \iota_j H' = -dB_{ij} \tag{2.12}$$

$$\iota_i H' = dv_i \tag{2.13}$$

and (2.11) implies

$$\mathcal{L}_i B_{jk} = 0 \tag{2.14}$$

The 1-forms v_i are only defined up to the addition of an exact 1-form. Consider then

$$v'_i = v_i - df_i \tag{2.15}$$

where f_i are functions on V satisfying

$$\iota_i df_j = B_{ij} + \iota_i v_j \tag{2.16}$$

The integrability condition $\iota_k d\iota_i df_j = \iota_i d\iota_k df_j$ for (2.16) is satisfied as a result of (2.11) [10], so that solutions f_i exist. Then

$$\iota_i v'_j = B_{ij} \tag{2.17}$$

and

$$\mathcal{L}_i v'_j = 0 \tag{2.18}$$

The locally-defined 1-forms

$$\tilde{A}_i = v'_i + B_{ij}\xi^j \tag{2.19}$$

are horizontal

$$\iota_i \tilde{A}_j = 0 \tag{2.20}$$

and invariant

$$\mathcal{L}_i \tilde{A}_j = 0 \tag{2.21}$$

so that they can be regarded as 1-forms on $U \subset N$. They are connections for a bundle over N [10] with curvature

$$\tilde{F}_i = d\tilde{A}_i \tag{2.22}$$

which is horizontal, $\iota_i \tilde{F}_j = 0$. Then

$$H = \bar{H} + \tilde{F}_i \wedge \xi^i + dB + K \tag{2.23}$$

where

$$B = \frac{1}{2} B_{ij} \xi^i \wedge \xi^j \tag{2.24}$$

and \bar{H} is a 3-form on N satisfying

$$d\bar{H} = -\tilde{F}_i \wedge F^i; \tag{2.25}$$

A 2-form potential b with $db = H$ is given by

$$b = \bar{b} + \xi^i \wedge \tilde{A}_i + \frac{1}{2} B_{ij} \xi^i \wedge \xi^j + \kappa \tag{2.26}$$

where $d\kappa = K$, so that κ can be taken to be

$$\kappa = \frac{1}{6} K_{ijk} X^i \wedge dX^j \wedge dX^k \tag{2.27}$$

\bar{b} is a 2-form on $U \subset N$ with

$$\bar{H} = d\bar{b} + F^i \wedge \tilde{A}_i \tag{2.28}$$

Using the symmetry, these results extend from a contractible patch to any patch of the form $U' = U \times T^d$.

In adapted local coordinates $\phi^\mu = (X^i, Y^m)$ in which

$$k_i^\mu \frac{\partial}{\partial \phi^\mu} = \frac{\partial}{\partial X^i} \tag{2.29}$$

the Lie derivative is the partial derivative with respect to X^i , so that $g_{\mu\nu}, H_{\mu\nu\rho}$ are independent of X^m . Then

$$\xi^i = dX^i + A^i \tag{2.30}$$

where $A^i = A_m^i(Y) dY^m$ satisfies $\iota_i A^j = 0$ and

$$dA^i = F^i \tag{2.31}$$

satisfies $\iota_i F^j = 0$. The A^i are connection 1-forms for M viewed as a bundle over N .

The derivation of T-duality of [10], generalising that of [17–20], involves the gauging of the symmetry generated by the k_i . If $K = 0$ and the \tilde{A} and the B_{ij} are globally defined, then the obstructions to gauging of [23] are absent as a result of (2.17), (2.18); however, in general \tilde{A} and B_{ij} will not be globally defined.

In general the $\tilde{A}_i = \tilde{A}_{im}(Y)dY^m$ are connections for a dual bundle \tilde{M} over N , built from patches $U \times \tilde{T}^d$, and so will not be globally defined. Globally defined one-forms are defined by introducing fibre coordinates \tilde{X}_i on \tilde{T}^d so that

$$\tilde{\xi}_i = d\tilde{X}_i + \tilde{A}_i \tag{2.32}$$

is a well-defined 1-form on \tilde{M} and

$$\tilde{F}_i = d\tilde{\xi}_i = d\tilde{A}_i \tag{2.33}$$

is also horizontal. To be able to define a well-defined quantum sigma-model, the fibres \tilde{T}^d are taken to be the torus dual to the torus fibres in $U' = U \times T^d$ [10]. If X^i has period $2\pi R^i$ and \tilde{X}^i has period $2\pi \tilde{R}_i$, then these are related by $R^i = \alpha' / \tilde{R}_i$. In addition to the k_i , there are vector fields \tilde{k}^i tangent to the new fibres

$$\tilde{k}^i = \frac{\partial}{\partial \tilde{X}_i} \tag{2.34}$$

The \tilde{k}^i commute with the k_i .

This allows the construction of a doubled patch $\hat{U} = U \times T^d \times \tilde{T}^d$ with fibres T^{2d} , coordinates Y^m and

$$\mathcal{X}^I = \begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix} \tag{2.35}$$

where $I = 1, \dots, 2d$, and connections

$$\mathcal{A}^I = \begin{pmatrix} A^i \\ \tilde{A}_i \end{pmatrix} \tag{2.36}$$

so that the one-forms

$$\Xi = \begin{pmatrix} \xi^m \\ \tilde{\xi}_m \end{pmatrix} \tag{2.37}$$

are well-defined 1-forms.

The isometries on \hat{M} can now be gauged provided there is no 3-flux on the fibres [10]:

$$\iota_i \iota_j \iota_k H = 0 \tag{2.38}$$

so that $H' = H$, and this will be assumed to be the case here. This is also the condition for conventional T-duality to be possible [10].

For geometric backgrounds, the patches $U' = U \times T^d$ patch together to give a manifold M , the dual patches $\tilde{U} = U \times \tilde{T}^d$ patch together to form a dual manifold \tilde{M} (the T-dual of M , again a torus bundle over N) and the $\hat{U} = U \times T^d \times \tilde{T}^d$ patch together to form a

manifold \hat{M} , which is a T^{2d} bundle over N . For T-folds, the U' or \tilde{U} may not patch to form manifolds, but \hat{M} is a well-defined T^{2d} bundle over N , a geometric space containing all the information about the background and all its T-duals. It is this well-defined manifold \hat{M} that is used to construct the string action for a T-fold background using the doubled formalism [1].

There is a natural action of $O(d, d)$ on \hat{U} and hence on \hat{M} . Consider $h \in O(d, d)$ given by

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.39}$$

where a, b, c, d are $d \times d$ matrices. This preserves the indefinite metric

$$L = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{2.40}$$

so that

$$h^t L h = L \Rightarrow a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = \mathbb{1}. \tag{2.41}$$

The group $O(d, d, \mathbb{Z})$ consists of matrices (2.39) with integral entries. Then $\Xi, \mathcal{X}, \mathcal{A}$ transform in the fundamental representation

$$\Xi \rightarrow \Xi' = h^{-1} \Xi \tag{2.42}$$

$$\mathcal{A} \rightarrow \mathcal{A}' = h^{-1} \mathcal{A}, \quad \mathcal{X} \rightarrow \mathcal{X}' = h^{-1} \mathcal{X} \tag{2.43}$$

Defining

$$E_{ij} = G_{ij} + B_{ij} \tag{2.44}$$

E transforms non-linearly under $O(d, d)$ [22, 19, 16, 10]

$$E' = (aE + b)(cE + d)^{-1}. \tag{2.45}$$

The moduli G, B can be used to define a natural metric on the fibres given by the $2d \times 2d$ matrix \mathcal{H}_{IJ} given by

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.46}$$

which transforms covariantly under $O(d, d)$

$$\mathcal{H} \rightarrow h^t \mathcal{H} h \tag{2.47}$$

Note that the G, B are well-defined moduli and are scalar fields on N , so that the metric (2.46) and the transformations (2.45) are well-defined. Similar formulae involving the components of the gauge field b are potentially problematic as b is only defined up to gauge transformations.

3. Doubled formalism

The doubled formalism [1] is based on the duality-covariant formalism of [30] (and similar to models of [21, 31–36]). It is $O(d, d; \mathbb{Z})$ covariant and written in a patch \hat{U} of \hat{M} in terms of $\mathbb{X}, \mathcal{A}, \mathcal{H}$. The usual formalism arises on choosing a polarisation, i.e. a choosing a physical subspace $U \times T^d \subset U \times T^{2d}$.

Consider a patch \hat{U} of a space \hat{M} which is a T^{2d} bundle over N , with fibre coordinates \mathbb{X}^I , local coordinates Y^m on $U \subset N$ and connection 1-forms

$$\mathcal{A}^I = \mathcal{A}_m^I dY^m \tag{3.1}$$

Let L_{IJ} be the constant $O(d, d)$ invariant metric (2.40) on the fibres, and let \mathcal{H}_{IJ} be a positive-definite fibre metric satisfying

$$L^{-1} \mathcal{H} L^{-1} \mathcal{H} = \mathbb{1} \tag{3.2}$$

This ‘generalised metric’ is assumed to be independent of \mathbb{X} but is a function $\mathcal{H}_{IJ}(Y)$ on N . Then

$$S^I{}_J = L^{IK} \mathcal{H}_{KJ} \tag{3.3}$$

satisfies

$$S^2 = \mathbb{1} \tag{3.4}$$

and so defines an almost product (or almost real) structure.

The sigma-model with target space \hat{M} of [1] is a theory of maps from a 2-dimensional world-sheet W to \hat{M} , given locally by $\mathbb{X}^I(\sigma)$ where σ^α are coordinates on W . The pull-back of $d\mathbb{X}^I$ gives the fibre momentum

$$\mathcal{P}_\alpha^I = \partial_\alpha \mathbb{X}^I \tag{3.5}$$

while the pull-back of the one-forms Ξ^I gives the covariant fibre momentum $\hat{\mathcal{P}}^I$, which is a 1-form on W with components

$$\hat{\mathcal{P}}_\alpha^I = \mathcal{P}_\alpha^I + \mathcal{A}_m^I \partial_\alpha Y^m \tag{3.6}$$

The lagrangian of [1] is

$$\mathcal{L}_d = \frac{1}{4} \mathcal{H}_{IJ} \hat{\mathcal{P}}^I \wedge * \hat{\mathcal{P}}^J - \frac{1}{2} L_{IJ} \mathcal{P}^I \wedge \mathcal{A}^J + \mathcal{L}(Y) \tag{3.7}$$

where $\mathcal{L}(Y)$ is the lagrangian for a sigma-model with target space N , and all forms have been pulled back to W . The unusual normalisation with a factor of 1/4 is important and needed to give equivalence with the canonically normalised standard sigma-model lagrangian (2.1). The Wess-Zumino term

$$S_{WZ} = -\frac{1}{2} \int_W L_{IJ} \mathcal{P}^I \wedge \mathcal{A}^J \tag{3.8}$$

can be rewritten as

$$S_{WZ} = -\frac{1}{2} \int_V L_{IJ} \mathcal{P}^I \wedge \mathcal{F}^J \tag{3.9}$$

where V is a 3-manifold with boundary W , and \mathcal{F}^I is the pull-back

$$\frac{1}{2}\mathcal{F}_{mn}^I\partial_\alpha Y^m\partial_\beta Y^n d\sigma^\alpha\wedge d\sigma^\beta$$

of the curvature

$$\mathcal{F}^I = d\mathcal{A}^I \tag{3.10}$$

In later sections, it will be useful to consider adding a topological term

$$\mathcal{L}_{\text{top}} = \frac{1}{2}\Omega_{IJ}d\mathcal{X}^I\wedge d\mathcal{X}^J \tag{3.11}$$

for some constant $\Omega_{IJ} = -\Omega_{JI}$. This does not contribute to the field equations and does not affect the classical theory, but plays a role in the quantum theory.

This theory is subjected to the constraint [1]

$$\hat{\mathcal{P}} = S * \hat{\mathcal{P}} \tag{3.12}$$

where $*$ is the Hodge dual on the world-sheet satisfying $(*)^2 = 1$ (assuming Lorentzian signature world-sheet; for W with Euclidean signature, the constraint is $\hat{\mathcal{P}} = -iS * \hat{\mathcal{P}}$.) If the sigma-model on N has a lagrangian

$$\mathcal{L}(Y) = \mathcal{L}'(Y) - A^i \wedge A_i \tag{3.13}$$

where

$$\mathcal{L}'(Y) = \frac{1}{2}\bar{g}_{mn}dY^m\wedge*dY^n + \frac{1}{2}\bar{b}_{mn}dY^m\wedge dY^n \tag{3.14}$$

for some $\bar{g}_{mn}(Y), \bar{b}_{mn}(Y)$ on the base N , then it was shown in [1] that the doubled sigma-model (3.7) with constraint (3.12) is classically equivalent to the conventional sigma-model (2.1) with metric (2.7) and 2-form (2.26). In section 6, this result will be re-derived and extended to the quantum theory in section 8.

The field equation from varying \mathcal{X}^I in (3.7) is

$$d*(\mathcal{H}_{IJ}\hat{\mathcal{P}}^J) = L_{IJ}\mathcal{F}^J \tag{3.15}$$

which can be rewritten as

$$d*(S^I{}_J\hat{\mathcal{P}}^J - *\hat{\mathcal{P}}^I) = 0 \tag{3.16}$$

so that the constraint (3.12) implies the field equation (3.15) (and is a stronger condition).

The lagrangian is manifestly invariant under the rigid $GL(2d, \mathbb{R})$ transformations

$$\mathcal{H} \rightarrow h^t \mathcal{H} h, \quad \mathcal{P} \rightarrow h^{-1} \mathcal{P}, \quad \mathcal{A} \rightarrow h^{-1} \mathcal{A} \tag{3.17}$$

(with Y and $\mathcal{L}(Y)$ invariant). The corresponding transformation of the coordinates

$$\mathcal{X} \rightarrow h^{-1} \mathcal{X} \tag{3.18}$$

only preserves the boundary conditions if g is restricted to be in the subgroup $GL(2d, \mathbb{Z}) \subset GL(2d, \mathbb{R})$ preserving the periodicities of the \mathcal{X} . The constraint (3.12) breaks $GL(2d, \mathbb{R})$ to the subgroup $O(d, d)$ preserving L^{IJ} and so breaks $GL(2d, \mathbb{Z})$ to $O(d, d; \mathbb{Z})$. Thus this formulation is manifestly invariant under the T-duality group $O(d, d; \mathbb{Z})$. The topological term (3.11) is invariant if $\Omega \rightarrow h^t \Omega h$ under these transformations.

4. Polarisation and T-duality

In order to make contact with the conventional formulation, one needs to choose a polarisation, i.e. to choose a splitting of T^{2d} into a physical T^d and a dual \tilde{T}^d for each point in N , splitting the fibre coordinates into the physical coordinates $X \in T^d$ and the dual coordinates $\tilde{X} \in \tilde{T}^d$, and then write the theory in terms of the coordinates X alone, solving the constraint (3.12) to express $\tilde{X}(\sigma)$ in terms of $X(\sigma)$. Then the variables X are the ones integrated over in the functional integral, and invariance of the theory under T-duality implies that the physics should be independent of the choice of polarisation.

In order to define a polarisation or local product structure on the fibres, one first chooses a subgroup $GL(d, \mathbb{R})$ of $O(d, d)$ under which the fundamental $\mathbf{2d}$ of $O(d, d)$ splits into the fundamental representation \mathbf{d} of $GL(d, \mathbb{R})$ and the dual representation \mathbf{d}' , $\mathbf{2d} \rightarrow \mathbf{d} \oplus \mathbf{d}'$. It will be useful to use a superscript i for the fundamental representation \mathbf{d} (where $i = 1, \dots, d$) and a subscript i for the dual representation \mathbf{d}' , and introduce constant projectors Π^i_I and $\tilde{\Pi}_{iI}$, so that

$$\mathcal{P} = \begin{pmatrix} \Pi^i_I \mathcal{P}^I \\ \tilde{\Pi}_{iI} \mathcal{P}^I \end{pmatrix} = \begin{pmatrix} P^i \\ Q_i \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} \Pi^i_I \mathcal{X}^I \\ \tilde{\Pi}_{iI} \mathcal{X}^I \end{pmatrix} = \begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix}, \quad (4.1)$$

with the X^i the coordinates of the T^d subspace and \tilde{X}_i the coordinates of the dual \tilde{T}^d subspace. This can be thought of as a choice of basis, but it is useful to introduce the projectors explicitly, so as to keep track of the choice of subgroup $GL(d, \mathbb{R})$ of $O(d, d)$; duality transformations change the projectors and change the subgroup $GL(d, \mathbb{R})$ to a conjugate one.

The metric L is off-diagonal in the $GL(d)$ basis and can be written as

$$L = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad (4.2)$$

so that the corresponding line element is

$$ds^2 = 2dX^i d\tilde{X}_i \quad (4.3)$$

Then the T^d submanifold with coordinates X^i is a maximally null subspace with respect to this metric. Choosing a polarisation that selects a maximal null $T^d \subset T^{2d}$ together with its complement \tilde{T}^d then corresponds to choosing a subgroup $GL(d, \mathbb{Z}) \subset O(d, d; \mathbb{Z})$.

It will be useful to introduce the notation \hat{I} for the $O(d, d)$ indices in the $GL(d)$ basis, so that for any vector v , $v^{\hat{I}} = (v^i, v_i)$ and the matrix giving the change from an arbitrary basis to the $GL(d)$ basis is

$$\Theta^{\hat{I}}_J = \begin{pmatrix} \Pi^i_J \\ \tilde{\Pi}_{iJ} \end{pmatrix} \quad (4.4)$$

with the corresponding matrix for the dual representation $\hat{\Theta} = L^{-1}\Theta L$ so that

$$\hat{\Theta}^{\hat{I}}_J = \begin{pmatrix} \tilde{\Pi}_{iJ} \\ \Pi^i_J \end{pmatrix} \quad (4.5)$$

where $\Pi^{iJ} = \Pi^i{}_I L^{IJ}$, $\tilde{\Pi}_i{}^J = \tilde{\Pi}_i{}_I L^{IJ}$. The matrix $\Theta^{\hat{I}}{}_J$ can be thought of as a representative of the coset $O(d, d)/GL(d, \mathbb{R})$, or as a ‘vielbein’ converting $O(d, d)$ indices to $GL(d)$ ones. Then the equations giving components in the $GL(d)$ basis can be rewritten as

$$\Theta \mathcal{P} = \begin{pmatrix} P^i \\ Q_i \end{pmatrix}, \quad \Theta \mathcal{A} = \begin{pmatrix} A^i \\ \tilde{A}_i \end{pmatrix} \quad (4.6)$$

The components of \mathcal{H} in this basis

$$\hat{\Theta} \mathcal{H} \hat{\Theta}^t = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \quad (4.7)$$

This notation will help in following the effects of changes of polarisation explicitly. In particular, (4.7) defines a metric G_{ij} and 2-form B_{ij} in terms of \mathcal{H} and a polarisation Θ .

The T-duality transformation rules $G \rightarrow G'$, $B \rightarrow B'$, $\mathcal{A} \rightarrow \mathcal{A}'$ (2.45), (2.43) are then obtained using the $O(d, d)$ transformations for \mathcal{H} , \mathcal{A} while keeping the polarisation Θ fixed,

$$\mathcal{H} \rightarrow \mathcal{H}' = h^t \mathcal{H} h, \quad \mathcal{A} \rightarrow \mathcal{A}' = h^{-1} \mathcal{A}, \quad \Theta \rightarrow \Theta' = \Theta \quad (4.8)$$

so that e.g.

$$\begin{aligned} G^{-1} = \Pi \mathcal{H} \Pi^t &\rightarrow (G')^{-1} = \Pi h^t \mathcal{H} h \Pi^t \\ BG^{-1} = \tilde{\Pi} \mathcal{H} \Pi^t &\rightarrow B'(G')^{-1} = \tilde{\Pi} h^t \mathcal{H} h \Pi^t \\ A = \Pi \mathcal{A} &\rightarrow A' = \Pi h^t \mathcal{A} \end{aligned} \quad (4.9)$$

These same transformations $G \rightarrow G'$, $B \rightarrow B'$, $\mathcal{A} \rightarrow \mathcal{A}'$ can also be obtained by keeping \mathcal{H} fixed while transforming Θ

$$\mathcal{H} \rightarrow \mathcal{H}' = \mathcal{H}, \quad \mathcal{A} \rightarrow \mathcal{A}' = \mathcal{A}, \quad \Theta \rightarrow \Theta' = \Theta h \quad (4.10)$$

so that

$$\Pi \rightarrow \Pi' = \Pi h \quad \tilde{\Pi} \rightarrow \tilde{\Pi}' = \tilde{\Pi} h \quad (4.11)$$

Thus the T-duality transformations can be viewed either as active transformations in which the geometry \mathcal{H} , \mathcal{A} is changed while $\Pi, \tilde{\Pi}$ are kept fixed (4.8), or as a passive one in which the geometry \mathcal{H} , \mathcal{A} is kept fixed but the polarisation is changed (4.10), (4.11). In the latter viewpoint, the doubled geometry is unchanged, but the choice of physical subspace is transformed. The symmetry under T-duality is then the statement that the physics does not depend on the choice of physical subspace.

5. Conserved currents

The one-forms on the world-sheet W

$$J_I = \mathcal{H}_{IJ} \hat{P}^J - L_{IJ} * \hat{P}^J \quad (5.1)$$

are conserved currents

$$d * J_I = 0 \quad (5.2)$$

(using the field equations (3.15)). It is the sum of a Noether current \mathcal{J}_I and the ‘topological’ current

$$j_I = L_{IJ} * \mathcal{P}^J \tag{5.3}$$

which is trivially conserved, $d * j = 0$ as $d\mathcal{P} = 0$. The Noether current is

$$\mathcal{J}_I = \mathcal{H}_{IJ} \hat{\mathcal{P}}^J - L_{IJ} * \mathcal{A}^J \tag{5.4}$$

where $\mathcal{A}^J = \mathcal{A}_m^J \partial_\alpha Y^m d\sigma^\alpha$ is the pull-back of \mathcal{A} , and this generates the symmetries

$$\delta \mathbb{X}^I = \alpha^I \tag{5.5}$$

of translation along the fibres. Note that J is gauge-invariant and so well-defined, while \mathcal{J} and j are not. The constraint (3.12) is $J_I = 0$. (Adding the topological term (3.11) would modify the Noether current by an identically conserved term, $\mathcal{J}_I \rightarrow \mathcal{J}_I + \Omega_{IJ} * \mathcal{P}^J$.)

Following [1], it is useful to introduce a $2d \times 2d$ vielbein $\mathcal{V}^A_I(Y)$ such that

$$\mathcal{H} = \mathcal{V}^t \mathcal{V} \tag{5.6}$$

with frame indices raised and lowered with δ_{AB} . There are then two metrics, \mathcal{H}_{IJ} with frame components δ_{AB} and L_{IJ} with frame components L_{AB} . They are both preserved by $O(d) \times O(d)$, and it is useful to choose a basis in which $O(d) \times O(d)$ is manifest. The indices $A, B = 1, \dots, 2d$ transform under $O(d) \times O(d)$ and can be split into indices $a, b = 1, \dots, d$ and $a', b' = 1, \dots, d$ for the two $O(d)$ factors, $A = (a, a')$, so that in a natural basis

$$L^{AB} = \begin{pmatrix} L^{ab} & 0 \\ 0 & L^{a'b'} \end{pmatrix} = \begin{pmatrix} \mathbb{1}^{ab} & 0 \\ 0 & -\mathbb{1}^{a'b'} \end{pmatrix}, \quad S^A_B = \begin{pmatrix} \delta^a_b & 0 \\ 0 & -\delta^{a'}_{b'} \end{pmatrix} \tag{5.7}$$

Then

$$\mathcal{V}^A_I = \begin{pmatrix} \mathcal{V}^a_I \\ \mathcal{V}^{a'}_I \end{pmatrix}, \quad \mathcal{V}^{\mathcal{P}} = \begin{pmatrix} \mathcal{P}^a \\ \mathcal{P}^{a'} \end{pmatrix}, \tag{5.8}$$

and

$$\mathcal{H}_{IJ} = \mathcal{V}^a_I \mathcal{V}^b_J \delta_{ab} + \mathcal{V}^{a'}_I \mathcal{V}^{b'}_J \delta_{a'b'} \tag{5.9}$$

The current

$$J^I = L^{IJ} J_J = S^I_J \hat{\mathcal{P}}^J - * \hat{\mathcal{P}}^I \tag{5.10}$$

has frame components $J^A = (J^a, J^{a'})$

$$\begin{aligned} J^a &= \hat{\mathcal{P}}^a - * \hat{\mathcal{P}}^a \\ J^{a'} &= \hat{\mathcal{P}}^{a'} + * \hat{\mathcal{P}}^{a'} \end{aligned} \tag{5.11}$$

The constraint (3.12) is $J = 0$ and this becomes

$$\begin{aligned} \hat{\mathcal{P}}^a &= + * \hat{\mathcal{P}}^a \\ \hat{\mathcal{P}}^{a'} &= - * \hat{\mathcal{P}}^{a'} \end{aligned} \tag{5.12}$$

Introducing null coordinates σ^\pm on the world-sheet, so that $\alpha = (+, -)$, these become

$$\begin{aligned}\hat{\mathcal{P}}_-^a &= 0 \\ \hat{\mathcal{P}}_+^{a'} &= 0\end{aligned}\tag{5.13}$$

while $J_\alpha^a, J_\alpha^{a'}$ are the chiral currents

$$\begin{aligned}J_+^a &= 0, & J_-^a &= \hat{\mathcal{P}}_-^a, \\ J_+^{a'} &= \hat{\mathcal{P}}_+^{a'}, & J_-^{a'} &= 0\end{aligned}\tag{5.14}$$

There are then two chiral currents, and their conservation law is

$$D * J^A = d * J^A - \omega^A_B \wedge * J^B = 0\tag{5.15}$$

where ω is the connection $\omega_\alpha = (\partial_\alpha \mathcal{V})\mathcal{V}^{-1}$ and has off-diagonal terms mixing the two currents. For example, the conservation law for J_-^a is

$$\partial_+ J_-^a - (\omega_+)^a_b J_-^b - (\omega_-)^a_{b'} J_+^{b'} = 0\tag{5.16}$$

Given a polarisation, one can define the currents

$$J^i = \Pi^i_I J^I\tag{5.17}$$

which are conserved $d * J^i = 0$ as Π^i_I is constant. The components of J_α^i are given, using (5.14), by

$$J_+^i = \Pi^i_{a'} \hat{\mathcal{P}}_+^{a'} \quad J_-^i = \Pi^i_a \hat{\mathcal{P}}_-^a\tag{5.18}$$

where

$$\Pi^i_a = \Pi^i_I \mathcal{V}^I_a, \quad \Pi^i_{a'} = \Pi^i_I \mathcal{V}^I_{a'}\tag{5.19}$$

As the matrices (5.19) are non-degenerate, $J^i = 0$ is equivalent to $J^I = 0$ as the only non-vanishing components of J^I are those in (5.14), and so $J^i = 0$ is equivalent to the constraint (3.12).

As before, $J^i = \mathcal{J}^i + j^i$ where $j^i = \Pi^i_I * d\mathcal{X}^I$ is trivially conserved and \mathcal{J}^i is the Noether current for the transformations

$$\delta \tilde{X}_i = \tilde{\alpha}_i, \quad \delta X^i = 0\tag{5.20}$$

Similarly, there are also conserved currents

$$J_i = \tilde{\Pi}_i^I J^I\tag{5.21}$$

with $J_i = 0$ equivalent to (3.12) and which generate the transformations $\delta X^i = \alpha^i, \delta \tilde{X}_i = 0$.

In the case of a trivial bundle with constant $\mathcal{H}_{IJ} = \delta_{IJ}$, the currents are

$$J^i = d\tilde{X}_i + *dX^i\tag{5.22}$$

(the flat metric can be used to identify upper and lower indices i, j and tangent space indices a, a'). The current $d\tilde{X}_i$ generates $\delta\tilde{X}_i = \alpha_i$ while $*dX$ is a topological current that is automatically conserved. Similarly, the current

$$J^i = dX^i + *d\tilde{X}_i \tag{5.23}$$

is the sum of a current dX^i generating $\delta X^i = \alpha^i$ and the topological current $*d\tilde{X}$. In the $O(d) \times O(d)$ basis

$$\mathcal{X}^I = \begin{pmatrix} X_R^a \\ X_L^{a'} \end{pmatrix} \tag{5.24}$$

with

$$X^i = \frac{1}{2}(X_L^i + X_R^i), \quad \tilde{X}_i = \frac{1}{2}(X_R^i - X_L^i) \tag{5.25}$$

Then

$$\mathcal{P}_\alpha^a = \partial_\alpha X_R^a, \quad \mathcal{P}_\alpha^{a'} = \partial_\alpha X_L^{a'} \tag{5.26}$$

and the currents (5.14) are

$$J_+^a = 0, \quad J_-^a = \partial_- X_R^a, \tag{5.27}$$

$$J_+^{a'} = \partial_+ X_L^{a'}, \quad J_-^{a'} = 0 \tag{5.28}$$

The symmetries generated by J^I are

$$\delta X_R^a = \alpha_R^a, \quad \delta X_L^{a'} = \alpha_L^{a'} \tag{5.29}$$

and J^i generates the anti-diagonal subgroup with $\alpha_L^i = -\alpha_R^i$ while J_i generates the diagonal subgroup with $\alpha_L^i = \alpha_R^i$. Note that the currents generate a Kac-Moody algebra

$$[J_-^a(\sigma), J_-^b(\sigma')] = d\delta^{ab}\delta'(\sigma - \sigma'), \quad [J_+^{a'}(\sigma), J_+^{b'}(\sigma')] = d\delta^{a'b'}\delta'(\sigma - \sigma'), \tag{5.30}$$

and so $J = 0$ is a second class constraint. This means that it cannot be imposed by adding a lagrange multiplier term $C \cdot J$, but might be imposed by supplementing this with a further term involving C^2 ; this will be discussed in the next section.

The constraint (3.12) then implies $\partial_- X_R^a = 0$ and $\partial_+ X_L^{a'} = 0$ so that X_R^a are right-movers and $X_L^{a'}$ are left-movers, giving the right count of degrees of freedom. The generalisation of this to the interacting case is that the constraint (3.12) implies that half of the currents J^I are chiral and the other half anti-chiral, but the projectors onto the chiral and anti-chiral parts change with the coordinate Y , as they are given in terms of $S(Y)$.

6. Imposing the constraint

The constraint (3.12) is $J^I = 0$ where J^I is the current (5.1). Given a polarisation, the constraint $J^i = 0$ where J^i is the current (5.17) also implies (3.12). A natural way of imposing the constraint is to attempt to gauge the symmetries generated by the current

J^I or J^i , as suggested in [1]. This involves introducing a gauge field C_I or C_i which is a one-form on the world-sheet. The linear Noether coupling is then

$$\frac{1}{2}C_I \wedge *J^I \tag{6.1}$$

or

$$\frac{1}{2}C_i \wedge *J^i \tag{6.2}$$

so that if this were the only term involving C , the gauge field would be a lagrange multiplier imposing the constraint $J = 0$. However, gauge invariance requires adding a term quadratic in C . Defining $C_A = (C_a, C_{a'})$ by $C_A = \mathcal{V}_A^I C_I$, using (5.14), the term (6.1) is

$$\frac{1}{2} \left(C_+^a J_-^a + C_-^{a'} J_+^{a'} \right) \tag{6.3}$$

and $C_-^a, C_+^{a'}$ do not appear, and as a result gives the same coupling as (6.2). However, there are in addition terms quadratic in C ; for the coupling to J^I , these do depend on $C_-^a, C_+^{a'}$, while for the coupling to J^i , they do not.

The first step in the gauging of J^I is given by minimal coupling, so that \mathcal{P}^I is replaced with

$$\mathcal{P}^I + L^{IJ} C_J \tag{6.4}$$

in the lagrangian (3.7) giving a gauge-invariant lagrangian. This gives a term linear in C of the form $C_I \wedge *j^I$ where $\mathcal{J}^I = J^I + *\mathcal{P}^I$, so that it differs from J by the identically conserved topological current $j^I = *\mathcal{P}^I$. The term (6.1) is then obtained by further adding a term

$$C_I \wedge *j^I = C_I \wedge \mathcal{P}^I \tag{6.5}$$

to the minimally-coupled action. However this term is not gauge-invariant and does not have a gauge-invariant completion. This is a case in which one of the obstructions to gauging of [23, 24] is present, and gauging is not possible.¹ If one ignores global issues and gauges the symmetry generated by \mathcal{J}_I in (3.7) to obtain a local lagrangian, there is a term quadratic in the gauge fields involving $C_-^a, C_+^{a'}$ as well as $C_+^a, C_-^{a'}$. As this is the gauging of the symmetry (5.5), this leads to the elimination of all the \mathcal{X}^I , leaving a sigma-model with fields Y on the base space N . Thus in this case, there is an obstruction to gauging with the currents J^I , so that the linear term (6.1) is not obtained, and if one gauges with the currents \mathcal{J}_I , then all of the \mathcal{X} are eliminated.

¹In the terminology of [23, 24], one is gauging the isometries generated by $2d$ Killing vectors k_I and the contraction of H with k_I is $\iota_I H = dv_I$, where v is determined up to exact terms. Choosing $v_I = \iota_I b$ and using the formulae of [23, 24] gives the gauging by minimal coupling. However, to obtain the coupling of the gauge field C to J instead of \mathcal{J} requires replacing v with $v'_I = v_I + L_{IJ} d^{-J}$, but now $\iota_I v'_J = L_{IJ}$ and the fact that these constants are non-zero implies that there is a local obstruction to gauging [23, 24]. However, while v' is a well-defined 1-form and J is a well-defined current, v and \mathcal{J} are only locally defined, so that the minimally-coupled action is not well-defined and there is a topological obstruction to the gauging. There is then an obstruction to gauging: v is not globally defined, while v' gives a non-zero $\iota_{(I} v'_{J)}$ and there is no v that overcomes both obstacles.

More interesting is the gauging of J^i . This takes the same form as (6.1) at the linearised level, but the quadratic term in the gauge fields just involves C_+^a, C_-^a , corresponding to gauging a diagonal subgroup of the gauge group for J^I . The gauged lagrangian is $\mathcal{L}_d + \mathcal{L}_g + \mathcal{L}_{\text{top}}$ where \mathcal{L}_d is the original lagrangian (3.7), \mathcal{L}_{top} is a topological term of the form (3.11) and

$$\mathcal{L}_g = \frac{1}{2}C_i \wedge *J^i + \frac{1}{4}\mathcal{H}^{ij}C_i \wedge *C_j \tag{6.6}$$

where

$$\mathcal{H}^{ij} = \Pi^i_I \Pi^j_J (\mathcal{H}^{-1})^{IJ} = \Pi^i_I \Pi^j_J (L^{-1})^{IK} \mathcal{H}_{KL} (L^{-1})^{LJ} \tag{6.7}$$

This gauged lagrangian can be derived as follows. Given a polarisation with

$$\mathcal{P} = \begin{pmatrix} \Pi^i_I \mathcal{P}^I \\ \tilde{\Pi}_i^I \mathcal{P}^I \end{pmatrix} = \begin{pmatrix} P^i \\ Q_i \end{pmatrix}, \quad \hat{\mathcal{P}} = \begin{pmatrix} \hat{P}^i \\ \hat{Q}_i \end{pmatrix} = \begin{pmatrix} P^i + A^i \\ Q_i + \tilde{A}_i \end{pmatrix} \tag{6.8}$$

the lagrangian (3.7) can be written as

$$\mathcal{L}_d = \frac{1}{4}G_{ij} \hat{P}^i \wedge * \hat{P}^j + \frac{1}{4}G^{ij} (\hat{Q}_i - B_{ik} \hat{P}^k) \wedge * (\hat{Q}_j - B_{jl} \hat{P}^l) - \frac{1}{2}(P^i \wedge \tilde{A}_i + Q_i \wedge A^i) + \mathcal{L}(Y) \tag{6.9}$$

The lagrangian (3.7) is a sigma-model on \hat{M} and the symmetry being gauged is (5.20), which can be viewed as an anti-diagonal subgroup of (5.5). Again, the first step is minimal coupling, corresponding to making the replacement

$$\mathcal{P}^I \rightarrow \mathcal{P}^I + C_i \Pi^i_J L^{IJ} \tag{6.10}$$

in (3.7) or equivalently to making the replacement

$$Q_i \rightarrow Q_i + C_i \tag{6.11}$$

in (6.9), giving a gauge-invariant lagrangian. This has a linear coupling

$$\frac{1}{2}C_i \wedge * \mathcal{J}^i, \quad \mathcal{J}^i = \Pi^i_j J^j = J^i - \Pi^i_I dX^I = J^i - P^i \tag{6.12}$$

to the Noether current \mathcal{J} , so that adding the term

$$\frac{1}{2}C_i \wedge P^i \tag{6.13}$$

coupling the gauge field C to the topological current $j^i = *P^i$ gives the linear coupling (6.2). In this case, the term (6.13) is gauge invariant up to a surface term, so that there is no local obstruction to the gauging.² However, this term is not invariant under large gauge transformations. An action invariant under large gauge transformations is given by adding the term

$$\mathcal{L}_{\text{top}} = \frac{1}{2}d\tilde{X}_i \wedge dX^i \tag{6.14}$$

²In this case, the potential obstruction to gauging is $\iota^i v^j = \Pi^i_I \Pi^j_J L^{IJ}$ and this vanishes identically.

which when added to (6.13) gives the term

$$\frac{1}{2}(d\tilde{X}_i + C_i) \wedge P^i \quad (6.15)$$

which is fully gauge-invariant under large gauge transformations. The term (6.14) corresponds to adding the topological term (3.11) to the classical lagrangian, with $\Omega_{IJ} = \tilde{\Pi}_{i[I}\Pi^i_{J]}$.

Defining

$$D_i = C_i + \hat{Q}_i - G_{ij} * \hat{P}^j - B_{ij} \hat{P}^j \quad (6.16)$$

the resulting lagrangian can be rewritten as

$$L = \frac{1}{2}G_{ij} \hat{P}^i \wedge * \hat{P}^j + \frac{1}{2}B_{ij} \hat{P}^i \wedge \hat{P}^j - \hat{P}^i \wedge A_i + L' \quad (6.17)$$

where

$$L' = \frac{1}{4}G^{ij}D_i \wedge *D_j + \mathcal{L}(Y) + A^i \wedge A_i \quad (6.18)$$

consists of an algebraic term for the D_i , which are then non-dynamical auxiliary fields and a term $\mathcal{L}'(Y) = \mathcal{L}(Y) + A^i \wedge A_i$ dependent only on Y and given by (3.13). In general coordinates,

$$L = \frac{1}{2}G_{ij}\xi_\mu^i \xi_\nu^j d\phi^\mu \wedge *d\phi^\nu + \left(\frac{1}{2}B_{ij}\xi_\mu^i \xi_\nu^j - \xi_\mu^i \tilde{A}_{i\nu} \right) d\phi^\mu \wedge d\phi^\nu + L' \quad (6.19)$$

Then, as was to be expected, the resulting theory is independent of \tilde{X} . Using (3.14), (3.13) it is precisely the original theory (2.1) with metric g given by (2.7) and b -field given by (2.26), plus the auxiliary field term D^2 . The invariance under large gauge transformations means that \tilde{X}_i can be completely gauged away, including winding modes, and this is reflected in the fact that the theory is independent of \tilde{X} after integrating out the gauge fields.

The term (6.14) is a topological term depending only on the winding numbers n^i, \tilde{n}_i of X^i, \tilde{X}_i around homology cycles in the world-sheet, so that it does not affect the classical theory. The periodicities of X, \tilde{X} are $2\pi R^i, 2\pi \tilde{R}_i$ with $\tilde{R}_i = \alpha'/R^i$ so that the T^d parameterised by the \tilde{X}_i is dual to the one parameterised by the X^i [19, 20, 10]. Then the term in the action $S = (2\pi\alpha')^{-1} \int \mathcal{L}_{\text{top}}$ is a sum of terms of the form $\pi n^i \tilde{n}_i$ (where n^i, \tilde{n}_i are winding numbers for a conjugate pair of cycles, and there is a sum over 1-cycles) and so contributes signs $e^{i\pi n^i \tilde{n}_i} = \pm 1$ to the functional integral given as a sum over winding numbers. A similar term arose in [19]. Note that changing the polarisation can change the sign of (6.14), but this leaves $e^{i\pi n^i \tilde{n}_i}$ unchanged, so does not change the quantum theory. For example, changing from the X^i polarisation to the \tilde{X}_i polarisation changes (6.14) by a factor of $(-1)^d$.

Thus the gauging gives back the original sigma-model (2.1). It can also be viewed as imposing the constraint $J = 0$. For example, choosing the gauge $C_- = 0$, C_+ becomes a lagrange multiplier imposing $J_- = 0$. Then the BRST constraints imply that J_+ annihilates physical states, so that in this way the full constraint $J_\pm = 0$ is achieved.

Thus given a polarisation, the constraint (3.12) can be realised by gauging the symmetry associated with the currents ΠJ , giving the conventional sigma-model (2.1). Different choices of polarisation give rise to different sigma-models and in each of these, half of the coordinates \mathbb{X} are gauged away. Different choices of polarisation select a different half of the coordinates \mathbb{X} and are related by $O(d, d; \mathbb{Z})$, and the different sigma-models obtained are all related by T-duality. For example, given a split $\mathbb{X} \rightarrow (X^i, \tilde{X}_i)$, choosing the polarisation as above gauges shifts in the \tilde{X}_i , giving a sigma-model with coordinates (Y, X) , while choosing the opposite polarisation gauges shifts in the X^i , giving the dual sigma-model with coordinates (Y, \tilde{X}) (corresponding to T-dualising all d circles).

7. T-folds

A T-fold is constructed from patches in each of which there is a conventional string background, but the patching conditions involve T-dualities, and in general lead to a non-geometric background. Let $\{U_\alpha\}$ be an open cover of the base N , $N = \cup_\alpha U_\alpha$.³ Then the T-fold is constructed from patches $U'_\alpha = U_\alpha \times T^d$, and in each such patch there is a metric g_α of the form (2.7) and a 2-form b_α of the form (2.26). The metric \bar{g}_α and 2-form \bar{b}_α on U_α are patched together in $U_\alpha \cap U_\beta$ using diffeomorphisms and b -field gauge transformations in the usual way. The remaining data specifying the geometry consists of the moduli $E_{ij}^\alpha = G_{ij}^\alpha + B_{ij}^\alpha$ and the $U(1)^{2d}$ connections $A_\alpha, \tilde{A}_\alpha$. Over overlaps $U_\alpha \cap U_\beta$, these are patched together using transition functions in $O(d, d; \mathbb{Z}) \times U(1)^{2d}$, where $O(d, d; \mathbb{Z})$ acts through (2.43), (2.45) and the $U(1)^{2d}$ acts through gauge transformations

$$\delta \mathcal{A}^I = d\Lambda^I, \quad \mathbb{X}^I = -\Lambda^I \tag{7.1}$$

This is a geometric background if the structure group is in the geometric subgroup $\Gamma(d, \mathbb{Z}) \times U(1)^{2d}$ where $\Gamma(d, \mathbb{Z}) = GL(d, \mathbb{Z}) \times \mathbb{Z}^{d(d-1)/2}$ is the group of large torus diffeomorphisms and integral shifts of B_{ij} . Otherwise, it is a T-fold [1].

Over each patch U_α one can instead consider a patch $U_\alpha \times T^{2d}$ with doubled fibre. As $O(d, d; \mathbb{Z}) \times U(1)^{2d}$ acts geometrically on T^{2d} , with $O(d, d; \mathbb{Z})$ acting as a subgroup of the large diffeomorphisms of T^{2d} , the T-fold transition functions in $O(d, d; \mathbb{Z}) \times U(1)^{2d}$ can be used for the patches $U_\alpha \times T^{2d}$ to construct a manifold \hat{M} as a T^{2d} bundle over N , with connection \mathcal{A} [1, 10]. In each patch one introduces a constant metric L_α of split signature (d, d) of the form (2.40) and a positive definite metric \mathcal{H}_α satisfying (3.2). The fibre metrics \mathcal{H}_α in each patch transform covariantly under $O(d, d)$ (3.17) and so have the transition functions

$$\mathcal{H}_\alpha = (h_{\alpha\beta})^t \mathcal{H}_\beta h_{\alpha\beta} \tag{7.2}$$

Similar transition functions for L are consistent with a constant $L_\alpha = L_\beta$ as the transition functions in $O(d, d; \mathbb{Z})$ preserve L .

Then for each patch, there is a doubled lagrangian \mathcal{L}_α given by (3.7), and in overlaps $\mathcal{L}_\alpha = \mathcal{L}_\beta$ so there is a well-defined action, which is a sigma-model with target space \hat{M} .

³In this section α, β will label coordinate patches and not world-sheet coordinates.

The constraint (3.12) is $O(d, d; \mathbb{Z})$ covariant, and so is a well-defined geometric condition for the sigma-model on \hat{M} .

One way of imposing this constraint is to choose a polarisation and gauge, as was shown in the last section. Consider first the case in which there are only $O(d, d; \mathbb{Z})$ transition functions, so

$$\mathbb{X}_\alpha^I = (h_{\alpha\beta}^{-1})^I{}_J \mathbb{X}_\beta^J \tag{7.3}$$

In each patch $U'_\alpha = U_\alpha \times T^d$, there is a choice of polarisation specified by projectors $\Pi_\alpha, \tilde{\Pi}_\alpha$, which can be combined into a matrix $(\Theta_\alpha)^{\hat{I}}{}_J$, as in section 4.1. This defines a splitting of the coordinates \mathbb{X}_α^I into ‘physical’ coordinates X_α^i and dual coordinates $\tilde{X}_{\alpha i}$

$$\mathbf{X}_\alpha^{\hat{I}} = (\Theta_\alpha)^{\hat{I}}{}_J \mathbb{X}_\alpha^J = \begin{pmatrix} (\Pi_\alpha)^i{}_I \mathbb{X}_\alpha^I \\ (\tilde{\Pi}_\alpha)_{iI} \mathbb{X}_\alpha^I \end{pmatrix} \tag{7.4}$$

where

$$\mathbf{X}_\alpha^{\hat{I}} \equiv \begin{pmatrix} X_\alpha^i \\ \tilde{X}_{\alpha i} \end{pmatrix}, \tag{7.5}$$

An active T-duality transformation transforms \mathbb{X} but leaves Θ invariant. Then the transition functions (7.3) will give an active T-duality transformation if the polarisation projector is constant, so that it is independent of the choice of patch

$$\Theta_\alpha = \Theta_\beta \tag{7.6}$$

Then in the overlap $U'_\alpha \cap U'_\beta$, the coordinates $\mathbf{X}_\alpha^{\hat{I}}$ are given by

$$\mathbf{X}_\alpha = \Theta_\alpha \mathbb{X}_\alpha = \Theta_\beta h_{\alpha\beta}^{-1} \mathbb{X}_\beta \tag{7.7}$$

The term $\Theta_\beta h_{\alpha\beta}^{-1} \mathbb{X}_\beta$ is regarded as arising from transition functions that are an active T-duality transforming \mathbb{X} , with $\Theta_\alpha = \Theta_\beta$, $\mathbb{X}_\alpha = h_{\alpha\beta}^{-1} \mathbb{X}_\beta$. The same \mathbf{X} could instead be regarded as arising from a passive T-duality acting on the polarisation with $\Theta_\alpha = \Theta_\beta h_{\alpha\beta}^{-1}$, but not on the coordinates, $\mathbb{X}_\alpha = \mathbb{X}_\beta$; in this section, the active viewpoint will be adopted, so that $\Theta_\alpha = \Theta$ is independent of the patch.

Then

$$\mathbf{X}_\alpha^{\hat{I}} = (\hat{h}_{\alpha\beta}^{-1})^{\hat{I}}{}_{\hat{J}} \mathbf{X}_\beta^{\hat{J}} \tag{7.8}$$

where

$$\hat{h}_{\alpha\beta} = \Theta h_{\alpha\beta} \Theta^{-1} \tag{7.9}$$

The matrix $\hat{h}_{\alpha\beta}$ has components

$$\hat{h}^{\hat{I}}{}_{\hat{J}} = \begin{pmatrix} \hat{h}^i{}_j & \hat{h}^{ij} \\ \hat{h}_{ij} & \hat{h}_i{}^j \end{pmatrix} \tag{7.10}$$

so that

$$X_\alpha^i = (\hat{h}_{\alpha\beta}^{-1})^i{}_j X_\beta^j + (\hat{h}_{\alpha\beta}^{-1})^{ij} \tilde{X}_{j\beta} \tag{7.11}$$

In each patch, the $\{X_\alpha^i\}$ are coordinates for a T^d fibre, and the condition for these to fit together to form a T^d bundle over N is that

$$(\hat{h}_{\alpha\beta}^{-1})^{ij} = 0 \tag{7.12}$$

so

$$X_\alpha^i = (\hat{h}_{\alpha\beta}^{-1})^i_j X_\beta^j \tag{7.13}$$

and the X_α^i are glued to the X_β^i . The condition (7.12) implies that the structure group is in the geometric subgroup $\Gamma(d, \mathbb{Z}) \subset O(d, d; \mathbb{Z})$, and implies that the T^d fibres are patched together with diffeomorphisms $(\hat{h}_{\alpha\beta}^{-1})^i_j \in GL(d, \mathbb{Z})$. Similarly, the dual tori \tilde{T}^d will fit together to form a bundle if

$$(\hat{h}_{\alpha\beta}^{-1})_{ij} = 0 \tag{7.14}$$

and the condition for there to be both a torus bundle with fibres T^d and a dual bundle with fibres \tilde{T}^d is that both (7.12) and (7.14) hold, so that the structure group is in $GL(d, \mathbb{Z})$.

This extends to the general case of a T-fold with structure group in $O(d, d; \mathbb{Z}) \times U(1)^{2d}$. In this case it is convenient to work with the $U(1)^{2d}$ -invariant 1-forms Ξ_α^I in each patch U'_α , with

$$\Xi_\alpha^{\hat{I}} = (\Theta_\alpha)^{\hat{I}}_J \Xi_\alpha^J = \begin{pmatrix} (\Pi_\alpha)^i_I \Xi_\alpha^I \\ (\tilde{\Pi}_\alpha)_{iI} \Xi_\alpha^I \end{pmatrix}, \tag{7.15}$$

where

$$\Xi_\alpha^{\hat{I}} \equiv \begin{pmatrix} \xi_\alpha^i \\ \tilde{\xi}_{\alpha i} \end{pmatrix}, \tag{7.16}$$

Then (7.12) is replaced with

$$\Xi_\alpha^{\hat{I}} = (\hat{h}_{\alpha\beta}^{-1})^{\hat{I}}_J \Xi_\beta^J \tag{7.17}$$

and so

$$\xi_\alpha^i = (\hat{h}_{\alpha\beta}^{-1})^i_j \xi_\beta^j + (\hat{h}_{\alpha\beta}^{-1})^{ij} \tilde{\xi}_{j\beta} \tag{7.18}$$

The condition that there is a T^d sub-bundle is that (7.12) holds, so that the structure group is in the geometric group $\Gamma(d, \mathbb{Z}) \times U(1)^{2d}$.

The currents J_α^I defined by (5.10) in each patch split into the currents $J_\alpha^i, \tilde{J}_{\alpha i}$ using the projectors $\Pi_\alpha, \tilde{\Pi}_\alpha$ and these have the transition functions

$$J_\alpha^i = (\hat{h}_{\alpha\beta}^{-1})^i_j J_\beta^j + (\hat{h}_{\alpha\beta}^{-1})^{ij} \tilde{J}_{j\beta} \tag{7.19}$$

Then the constraint $J_\alpha^i = 0$ is consistent with $J_\beta^i = 0$ only if (7.12) holds, so that the structure group is in the geometric group $\Gamma(d, \mathbb{Z}) \times U(1)^{2d}$. If this is the case, then the constraint (3.12) can be imposed by gauging by coupling J^i to gauge fields C_i . Note that if there are non-trivial $\Gamma(d; \mathbb{Z})$ transition functions, then the gauge fields C are not connections on a principle bundle, but instead are connections on the affine bundle given by the pull-back of \hat{M} to the world-sheet, with transition functions in $\Gamma(d; \mathbb{Z}) \times U(1)^d$ [10]. This is sufficient to give a well-defined gauged action, even though there are no globally-defined Killing vectors [10].

The bundle \hat{M} over N is characterised by the $2d$ first Chern classes, and the $O(d, d; \mathbb{Z})$ monodromies round the 1-cycles of N . If all monodromies are in a subgroup $\mathcal{M} \subseteq O(d, d; \mathbb{Z})$, then the structure group is in $\mathcal{M} \times U(1)^{2d}$. The lagrangian (3.7) is well-defined on \hat{M} , as is the constraint (3.12). The constraint (3.12) can be imposed by choosing a constant polarisation projector Π , with the same choice for each patch U_α , $\Pi_\alpha = \Pi_\beta$, and then gauging the current $J_\alpha^i = \Pi^i_I J_\alpha^I$ in each patch. The gauged lagrangians only patch together to give a well-defined action on \hat{M} if $\mathcal{M} \subseteq \Gamma(d; \mathbb{Z})$, so that the monodromies are all in the geometric subgroup, and in this case a geometric background is obtained. For non-geometric T-folds with monodromies not in the geometric group, there is no globally consistent choice of a physical T^d with coordinates X^i , and this is reflected in the fact that the gauged lagrangians in each U'_α do not patch together to form a well-defined classical lagrangian on \hat{M} . In the general case, the best one can do is to perform a different gauging in each patch. These do not then fit together to form a well-defined classical action. However, the patching is with a symmetry of the quantum theory, and the corresponding quantum theories do patch together to give a well-defined theory, as will be discussed in the next section.

8. Quantisation

In this section, the quantisation of a sigma-model on a T-fold is addressed. Suppose first the world-sheet W is flat. For the conventional formulation in terms of a sigma-model (2.1) with coordinates X^i, Y^m , one can first integrate over X . For a point $Y \in N$, the X are coordinates on a torus T^d and quantising the X gives the the standard torus CFT on T^d with moduli $G_{ij}(Y), B_{ij}(Y)$. CFT's with moduli related by $O(d, d; \mathbb{Z})$ transformations are equivalent, so that $O(d, d; \mathbb{Z})$ is a symmetry of the CFT, and the moduli space is not the coset $O(d, d)/O(d) \times O(d)$ parameterised by G, B , but is the Narain moduli space given by the quotient of this space by the action of $O(d, d; \mathbb{Z})$. Then the T-fold transition functions give a bundle of torus CFT's over N , and this is well-defined as the transition functions are a CFT symmetry.

The conformal field theory on T^d can also be formulated in an $O(d, d; \mathbb{Z})$ covariant way in terms of the doubled coordinate \mathbb{X} , imposing canonical commutation relations on \mathbb{X} and its conjugate momentum. However, in this approach one must also impose the constraint (3.12) and the issue arises as to how to impose this in the quantum theory. As has been seen, this can be done by choosing a polarisation and gauging the action of the current $J^i = \Pi^i_I J^I$. In general there will not be a global polarisation, and one must be chosen for each patch in N . One can then quantise in each patch to obtain the same torus CFT as before and these patch together to give the bundle of torus CFT's over N .

The final stage in the quantisation is then to integrate over the Y . The quantum theory for each patch from integrating over both Y and X is then the quantisation of the gauged sigma-model on $U_\alpha \times T^{2d}$. The ungauged action (3.7) is a sigma-model with target space $U' = U_\alpha \times T^{2d}$ and is renormalizable, as is the corresponding gauged model. The quantisation in the patch involves a choice of polarisation, but different choices lead to the

same quantum theory, and can be thought of as arising from T-dual versions of the same sigma-model.

The classical lagrangian (3.7) is globally well-defined on \hat{M} and is duality invariant, as is the constraint (3.12). The quantisation involves choosing a polarisation that selects the independent variables to be quantised and this breaks the duality symmetry and in general there is no global choice of polarisation. However, the quantum theory is duality invariant, and as the patching conditions involve a quantum symmetry, then the resulting quantum theory should be well-defined. It would be interesting to consider other ways of handling the constraint (3.12) in the quantum theory, and to compare the results.

Finally, in each patch, it has been seen that the two theories defined by the conventional sigma-model (2.1) and by gauging the doubled sigma-model (3.7) are classically equivalent, and each is quantisable, so the question arises as to whether they define the same quantum theory. To quantise the gauged model, one must first gauge-fix. With the topological term (6.14), the gauged action is invariant under gauge transformations, including large gauge transformations specified by maps from W to $U(1)^d$ with non-trivial monodromy around 1-cycles in W . These can be fixed by gauging \tilde{X} away completely, using the large gauge transformations to gauge away the winding modes of \tilde{X} . As was seen in section 6, this gives the conventional lagrangian (2.1), plus the auxiliary field term

$$\frac{1}{4}G^{ij}D_iD_j \tag{8.1}$$

In addition, there is a ghost term $b^i c_i$ where b^i, c_i are anti-commuting scalars. The ghost integration is trivial, so the result is the sum of (2.1) and (8.1), so that quantising the doubled formalism in this way is equivalent to the quantisation of a conventional sigma model (2.1) plus the auxiliary term (8.1). The auxiliary field term does not affect the classical dynamics, but as the matrix G^{ij} depends on the fields Y , integrating out D_i give a determinant that affects the functional measure for Y . It will be seen in the next section that this change in the measure can be absorbed into a shift of the dilaton, and that this is precisely what is needed to get the the correct dilaton coupling and transformation rules for the conventional sigma-model.

In this way one can define a quantum field theory for any T-fold geometry. It remains to impose the condition that these give modular invariant conformal field theories, and this requires imposes ‘field equations’ restricting the allowed backgrounds.

9. The dilaton coupling

For curved world-sheets, one can add to the doubled sigma-model action given by the integral of (3.7) the Fradkin Tseytlin term

$$S_{FT} = \int d^2\sigma \sqrt{h} \phi R \tag{9.1}$$

where R is the Ricci scalar for the world-sheet metric $h_{\alpha\beta}$, with $h = |\det(h_{\alpha\beta})|$ and ϕ is a scalar field on \hat{M} . It will be taken to be independent of the coordinates X, \tilde{X} so that it is

a function $\phi(Y)$ on N . It is then invariant under the $O(d, d; \mathbb{Z})$ symmetry of the doubled action.

On gauging and eliminating the gauge fields as in section 6, one must integrate over the auxiliary fields D_i with lagrangian

$$\frac{1}{4} G^{ij} D_i \wedge *D_j \tag{9.2}$$

Formally this gives a determinant involving $\Pi_\sigma \det(G_{ij}(X(\sigma)))$. If this is calculated as in [17, 39, 40], it gives a contribution to the Fradkin-Tseytlin term at one loop corresponding to replacing ϕ in (9.1) with

$$\Phi = \phi - \frac{1}{2} \log \det(G_{ij}) = \phi + \frac{1}{2} \log \det(\Pi \mathcal{H} \Pi^t) \tag{9.3}$$

so that the sigma-model action on M is the sum of the integral of (2.1) and the Fradkin-Tseytlin term

$$S_{FT} = \int d^2\sigma \sqrt{h} \Phi R \tag{9.4}$$

Under a T-duality

$$G^{-1} = \Pi \mathcal{H} \Pi^t \quad \rightarrow \quad (G')^{-1} = \Pi h^t \mathcal{H} h \Pi^t \tag{9.5}$$

and

$$\Phi \rightarrow \Phi' = \Phi + \frac{1}{2} \log \frac{\det G'}{\det G} \tag{9.6}$$

In this way, the standard T-duality transformations of the dilaton Φ are obtained. There are then two dilatons, related by (9.3). The dilaton Φ is the familiar one coupling to the conventional sigma-model through the term (9.4), transforming under T-duality as (9.6) and appearing as a scalar in the standard space-time effective actions. The dilaton ϕ coupling to the doubled sigma-model through (9.1) is invariant under $O(d, d; \mathbb{Z})$ and so T-duality is a symmetry of the perturbation theory in the coupling constant given by the expectation value of $e^{-\phi}$, but not of that defined by the expectation value of $e^{-\Phi}$. The expectation value of $e^{-\phi}$ is the string field theory coupling constant of [21]; see e.g. [37] for further discussion. There will be further corrections to the relation between the two dilatons arising in this way from higher loop contributions to the change in measure [40].

10. Doubled everything

The doubled formulation doubles the fibre coordinates X but not the base coordinates Y . A more democratic and covariant formulation would be to double the Y as well. This can always be done by adding some new coordinates \tilde{Y}_m and then gauging the shift symmetry $\delta \tilde{Y}_m = \tilde{\alpha}_m$, or more covariantly by imposing a constraint similar to (3.12) that can be imposed by such a gauging. The Y^m, \tilde{Y}_m are coordinates on some manifold \hat{N} . If N were a torus, the \tilde{Y} could be taken as coordinates on the dual torus, but for general N there is no obvious choice of a dual space for N . To generalise the preceding structure it is natural to demand that the tangent space $T\hat{N} \simeq (T \oplus T^*)N$ at each point, so that there is a natural

action of $O(n, n)$ on $T\hat{N}$, where n is the dimension of N . This suggests taking \hat{N} to be the cotangent bundle T^*N , or a quotient of this.

For general M of dimension D , we then double the coordinates ϕ^μ to obtain

$$\Phi^M = \begin{pmatrix} \phi^\mu \\ \tilde{\phi}_\mu \end{pmatrix} \tag{10.1}$$

which can be coordinates on T^*M or a quotient of this. If M is a T^d bundle over N , then \hat{N} can be taken to be a T^{2d} bundle over T^*N , which can be thought of as a quotient of T^*M in which the coordinates \tilde{X} (parameterising the fibres cotangent to T^d) are periodically identified. (In this section, Φ, ϕ are coordinates, not dilatons.) For the sigma model (2.1), we introduce a constant $O(D, D)$ invariant metric L_{MN} and a generalised metric \mathcal{G}_{MN} satisfying

$$\mathcal{S}^2 = \mathbb{1} \tag{10.2}$$

where

$$\mathcal{S} = L^{-1}\mathcal{G} \tag{10.3}$$

The doubled sigma model corresponding to (3.7) is then

$$\mathcal{L} = \frac{1}{4}\mathcal{G}_{MN}\mathcal{P}^M \wedge *\mathcal{P}^N \tag{10.4}$$

where

$$\mathcal{P}_\alpha^M = \partial_\alpha \Phi^M \tag{10.5}$$

This is subject to the constraint

$$\mathcal{P} = \mathcal{S} * \mathcal{P} \tag{10.6}$$

(As there are no undoubled coordinates, there is no connection \mathcal{A} .)

The constraint (10.6) can now be handled as in section 6. There is a natural polarisation in which the coordinates ϕ^μ of M are selected, using a projector Π^μ_M , as the real coordinates and the coordinates of the cotangent fibres $\tilde{\phi}_\mu$ are taken as auxiliary. In this polarisation in which Φ is given in terms of $\phi, \tilde{\phi}$ by (10.1), then (10.2) implies \mathcal{G} is of the form

$$\mathcal{G} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}. \tag{10.7}$$

for some symmetric $g_{\mu\nu}$ and anti-symmetric $b_{\mu\nu}$. The constraint (10.6) is equivalent to $J^\mu = 0$ where

$$J^\mu = \Pi^\mu_M J^M, \quad J^M = (\mathcal{S}\mathcal{P} - *\mathcal{P})^M \tag{10.8}$$

The constraint $J^\mu = 0$ can be imposed by coupling to gauge fields as in section 6, which involves gauging the shift symmetry $\delta\tilde{\phi} = \tilde{\alpha}$ generated by J^μ , and eliminating the gauge fields and the coordinates $\tilde{\phi}$ gives precisely the original lagrangian (2.1) (plus a topological term), by a similar argument to that given in section 6. Alternatively, if M is a T^d bundle over N , then J^μ decomposes into J^i, J^m and one can first impose the constraint $J^m = 0$ by coupling to gauge fields, and so gauge the shift symmetry $\delta\tilde{Y} = \beta$ generated by J^m .

This eliminates the \tilde{Y} and the doubled formalism lagrangian $\mathcal{L}(X, \tilde{X}, Y)$ (3.7) is recovered, with the remaining constraint $J^i = 0$.

For general T^*M , the generalised metric $\mathcal{G}_{MN}(\phi)$ depends only on the ϕ , not the $\tilde{\phi}$, so that $g(\phi), b(\phi)$ given by (10.7) are defined on M . Suppose that in a patch of M there are d commuting Killing vectors, so that one can choose adapted coordinates $\phi^\mu = (Y^m, X^i)$ so that the Killing vectors are $\partial/\partial X^i$ ($i = 1, \dots, d$); at this stage, no assumptions are made about whether or not the X^i are periodic. Then the lagrangian is invariant under shifts of $X^i, \tilde{X}_i, \tilde{Y}_m$ and under $GL(2d + n, \mathbb{R})$ acting as a linear transformation on the coordinates $X^i, \tilde{X}_i, \tilde{Y}_m$ (with $n = D - d$) and on \mathcal{G} by transformations similar to (3.17), (3.18). (Linear transformations involving the Y^m will not be a symmetry in general if \mathcal{G} depends non-trivially on the Y^m .) This is broken to $O(d + n, d)$ by the constraint (10.6), and if $d' \leq d$ of the X, \tilde{X} are periodic, then the boundary conditions further break the symmetry to the group $O(d', d'; \mathbb{Z}) \times O(d - d', d - d' + n)$. The discrete subgroup $O(d', d'; \mathbb{Z})$ is a gauge symmetry of the quantum theory (provided the $2d'$ periodic coordinates have the correct periodicities) with sigma-models related by the action of $O(d', d'; \mathbb{Z})$ giving equivalent quantum theories. As before, this can be thought of as changing the polarisation, so that it changes the d' -dimensional subset of the $2d'$ periodic coordinates that are to be physical. (Changing the polarisation for the non-periodic directions is not in general a gauge symmetry.)

11. Supersymmetry

As stated in [1], the supersymmetrisation of the doubled formalism is straightforward: the sigma-model (3.7) is replaced by a supersymmetric one. (The supersymmetric model was also discussed in [13].) The N=1 supersymmetric generalisation of (2.1) in (1,1) superspace is [38]

$$S = \frac{1}{2} \int d^2\sigma d^2\theta (g_{\mu\nu} C^{rs} + b_{\mu\nu} \gamma^{rs}) D_r \phi^\mu D_s \phi^\nu \quad (11.1)$$

where $\phi^\mu(\sigma, \theta)$ is a superfield on the superspace world-sheet with coordinates σ^α, θ^r where θ^r are real anti-commuting coordinates transforming as a world-sheet spinor, $r = 1, 2$ is a world-sheet spinor index, and D_r are the usual supercovariant derivatives. Here $C^{rs} = \epsilon^{rs}$ is the charge conjugation matrix and $\gamma^{rs} = C^{rt}(\gamma_3)_t^s = \gamma^{sr}$ where $(\gamma_3)_t^s$ is the chirality operator satisfying $(\gamma_3)^2 = 1$. The N=1 supersymmetric generalisation of (3.7) in (1,1) superspace is the superspace lagrangian

$$\mathcal{L}_s = \frac{1}{4} \mathcal{H}_{IJ} C^{rs} \hat{\mathcal{P}}_r^I \hat{\mathcal{P}}_s^J - \frac{1}{2} \gamma^{rs} L_{IJ} \mathcal{P}_r^I \mathcal{A}_s^J + \mathcal{L}(Y) \quad (11.2)$$

where $\mathcal{X}(\sigma, \theta), Y(\sigma, \theta)$ are now superfields,

$$\mathcal{P}_r^I = D_r \mathcal{X}^I, \quad \hat{\mathcal{P}}_r^I = \mathcal{P}_r^I + \mathcal{A}_m^I D_r Y^m \quad (11.3)$$

The superspace versions of (3.13), (3.14) are

$$\mathcal{L}(Y) = \mathcal{L}'(Y) - \gamma^{rs} A_r^i A_{si} \quad (11.4)$$

and

$$\mathcal{L}'(Y) = \frac{1}{2} (\bar{g}_{mn} C^{rs} + \bar{b}_{mn} \gamma^{rs}) D_r Y^m D_s Y^n \quad (11.5)$$

The supersymmetric version of the topological term (3.11) is

$$\mathcal{L}_{\text{top}} = \frac{1}{2} \Omega_{IJ} \gamma^{rs} \mathcal{P}_r^I \mathcal{P}_s^J \quad (11.6)$$

and the component expansion of this gives the topological term (3.11) plus the total derivative of a fermion bilinear.

The constraint (3.12) becomes

$$\hat{\mathcal{P}} = S\gamma_3 \hat{\mathcal{P}} \quad (11.7)$$

The component expansion gives fermionic bilinear contributions to the constraint (3.12), and a constraint on the world-sheet fermions ψ^I which reduces to $\psi = S\gamma_3 \psi$ in the free case, so that ψ^a is a left-handed chiral spinor and $\psi^{a'}$ is a right-handed one.

As in the bosonic case, this can be imposed by choosing a polarisation and gauging as in section 6, coupling to a superspace gauge field Γ_{ri} . The superspace current J_r^i corresponding to (5.10) is

$$J_r^i = \Pi^i_I J_r^I \quad (11.8)$$

where

$$J_r^I = L^{IJ} J_{rJ} = S^I_J \hat{P}_r^J - (\gamma_3)_r^s \hat{P}_s^I \quad (11.9)$$

The supersymmetric gauging is given by adding to (11.2) the supersymmetric generalisation of (6.6) given by

$$\mathcal{L}_g = \frac{1}{2} C^{rs} \Gamma_{ri} J_s^i + \frac{1}{4} \mathcal{H}^{ij} C^{rs} \Gamma_{ri} \Gamma_{sj} \quad (11.10)$$

Then eliminating the gauge field and \tilde{X}_i as in section 6, one recovers the lagrangian (11.1), giving the local equivalence of the formalisms. The discussion of quantisation and global structure extend straightforwardly to the supersymmetric case.

The formulation of section 10 also generalises straightforwardly to superspace giving the superspace lagrangian

$$\mathcal{L} = \frac{1}{4} \mathcal{G}_{MN} C^{rs} \mathcal{P}_r^M \mathcal{P}_s^N \quad (11.11)$$

subject to the constraint

$$\mathcal{P} = S\gamma_3 \mathcal{P} \quad (11.12)$$

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